A New Approach for Evaluating the Error Probability in the Presence of Intersymbol Interference and Additive Gaussian Noise

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The determination of the error probability of a data transmission system in the presence of intersymbol interference and additive gaussian noise is a major goal in the analysis of such systems. The exhaustive method for finding the error probability calculates all the possible states of the received signal using an N-sample approximation of the true channel impulse response. This method is too time-consuming because the computation involved grows exponentially with N. The worst-case sequence bound avoids the lengthy computation problem but is generally too loose.

In this paper, we have developed a new method which yields the error probability in terms of the first 2k moments of the intersymbol interference. A recurrence relation for the moments is derived. Therefore, a good approximation to the error probability of the true channel can be obtained by choosing N large enough, and the amount of computation involved increases only linearly with N. The series expansion is shown to be absolutely convergent, and an upper bound on the series truncation error is given. In order to show the improvement provided in this new method, it is compared with the Chernoff bound technique in three representative cases. An order of magnitude improvement in accuracy is obtained.*

I. INTRODUCTION

An important problem in the analysis of binary digital data systems is the determination of the system performance in the presence of intersymbol interference and additive gaussian noise. Since it is usually the most meaningful criterion in designing a digital data

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* In April 1970, the authors were advised by R. W. Pulleyblank that a similar method was discovered independently by M. Celebiler and O. Shimbo to be presented in a paper which will be published in Conference Record, ICC, 1970.

system, the error probability is chosen as the measure of the system performance.

Two alternatives are available at present. The first alternative[1,2] considers a truncated N-pulse-train approximation of the true channel. The error probability is calculated by evaluating the conditional error probability of each of 2^N possible data sequences and averaging over all 2^N sequences. Since each calculation of the conditional error probability takes a great deal of computer time, the number of sequences must be held to several thousand.[3] This limitation leads to a poor approximation of the true channel, and the error probability so obtained is not very useful. The second alternative evaluates an upper bound of the error probability by either the worst-case sequence[3] or the Chernoff inequality.[4,5] In many cases, the bound is too loose.

In this study we have developed a new way to evaluate the error probability in terms of the first $2k$ moments of the intersymbol interference. It provides a significant improvement in accuracy over the worst-case sequence bound or the Chernoff bound. The computations increase only linearly with N. Thus a good approximation of the true channel may be obtained. The convergence of this alternative is proved. Throughout, additive gaussian noise and independence of information digits are assumed. The generalization to a multilevel system is straightforward; hence, only binary systems will be considered in this study.

II. BRIEF DESCRIPTION OF THE SYSTEM

A simplified block diagram of a binary amplitude modulation (AM) data system is shown in Fig. 1. We assume that a single $s(t)$ having amplitude a_ℓ is transmitted through the channel every T seconds. The system transfer function is

$$R(\omega) = S(\omega)T(\omega)E(\omega) \tag{1}$$

where $s(t)$ and $r(t)$ are the Fourier transform pair of $S(\omega)$ and $R(\omega)$, respectively. In the absence of channel noise, a sequence of input channel signals

$$\sum_{\ell=-\infty}^{\infty} a_\ell s(t - \ell T), \tag{2}$$

will generate a corresponding output sequence

$$\sum_{\ell=-\infty}^{\infty} a_\ell r(t - \ell T), \tag{3}$$

where $\{a_\ell\}$ is a sequence of independent binary random variables,

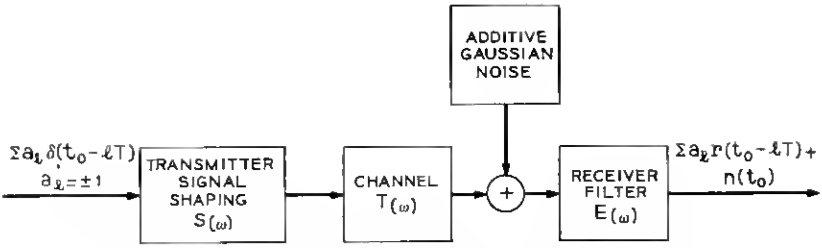

Fig. 1—Simplified block diagram of a binary AM data system.

$a_\ell = \pm 1$, and satisfies

$$P_r(a_\ell = 1) = P_r(a_\ell = -1) = \tfrac{1}{2}$$

$$\ell = -\infty, \cdots, -1, 0, 1, \cdots \infty. \tag{4}$$

We also assume that additive gaussian noise is present in the system. Thus the corrupted received sequence at the input to the receiver detector is

$$y(t) = \sum_{\ell=-\infty}^{\infty} a_\ell r(t - \ell T) + n(t), \tag{5}$$

where $n(t)$ is additive gaussian noise with a one-sided power spectral density of σ^2 watts/cps.

At the detector, $y(t)$ is sampled every T seconds to determine the transmitted signal. At sampling instant t_0, the sampled signal is

$$y(t_0) = a_0 r(t_0) + \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} a_\ell r(t_0 - \ell T) + n(t_0). \tag{6}$$

The first term is the desired signal while tbe second and the third terms represent the intersymbol interference and gaussian noise respectively.

It is well known that the optimum (minimum error probability) decision level is zero. Thus tbe error probability is given by

$$P_e = P_r\left\{\left[\sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} a_\ell r(t_0 - \ell T) + n(t_0)\right] \geq r(t_0)\right\}. \tag{7}$$

For the real system we are interested in, we may assume tbat the $a_\ell r(t_0 - \ell T)$'s are uniformly bounded and $\sum_{\ell \neq 0} a_\ell r(t_0 - \ell T)$ converges absolutely.* For example, in a system baving an open binary eye,

* Finite truncated pulse-train approximation will be used for those pulses with absolutely divergent intersymbol interference.

$\sum_{\ell \neq 0} | r(t_0 - \ell T) |$ is less than $r(t_0)$. Tbus by Kolmogorov's Tbree-Series criterion,[6] it can be easily sbown tbat $\sum_{\ell \neq 0} a_\ell r(t_0 - \ell T)$ converges absolutely to a random variable.

Equation (7) can be calculated by evaluating the expected value of the conditional expectation of tbe error probability for a given random variable $\sum_{\ell \neq 0} a_\ell r(t_0 - \ell T)$; tberefore,

$$P_e = \int_{\text{all } X} \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{0} \exp\left[-\{y - r(t_0) - X\}^2/2\sigma^2\right] dy\, dF(x), \tag{8}$$

where $F(X)$ is the distribution function of the random variable X, and $X = \sum_{\ell \neq 0} a_\ell r(t_0 - \ell T)$.

III. SERIES EXPANSION OF P_e

With the exception of a few special cases, equation (8) is generally difficult to solve. The existing solutions are either too time-consuming[1,2] or inaccurate.[3,4,5]

We have found that equation (8) can be evaluated in terms of an absolutely convergent series involving moments of the intersymbol interference. Furthermore, the moments can be obtained readily through recurrence relations. Therefore, the computation time is significantly reduced in comparison with the exhaustive method.[1,2] The absolute convergence and the recurrence relations for the moments are given in Appendix A and B respectively.

Expanding equation (8), we obtain the following expression for the error probability,

$$\begin{aligned} P_e &= \tfrac{1}{2} \operatorname{erfc}\left(-\frac{r(t_0)}{\sqrt{2}\,\sigma}\right) + \sum_k \frac{1}{(2k)!} \cdot \left(\frac{1}{2\sigma^2}\right)^k \cdot M_{2k} \cdot \frac{1}{\sqrt{\pi}} \\ &\quad \cdot \left[\exp - \left(\frac{r^2(t_0)}{2\sigma^2}\right)\right] \cdot H_{2k-1}\left(\frac{r(t_0)}{\sqrt{2}\,\sigma}\right) \qquad k = 1, 2, 3, \cdots, \\ &= P_{e_0} + \sum_{k=1}^{\infty} P_{e_{2k}}, \end{aligned} \tag{9}$$

where $H_{2k-1}(x)$ is a Hermite polynominal, M_{2k} is the $2k$th moment of the random variable X, and

$$\operatorname{erfc}(-x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{-x} \exp(-z^2)\, dz. \tag{10}$$

The first term in equation (9) represents the nominal system error probability due to additive gaussian noise alone while the summation

represents the degradation of the system performance due to intersymbol interference in the additive gaussian noise environment.

3.1 *Convergence Property*

In Appendix A we have shown that equation (9) is an absolutely convergent series. Therefore, the error probability can be evaluated by taking a finite number of terms,

$$P_e = \sum_{k=0}^{K-1} P_{e_{2k}} + R_{2K} , \tag{11}$$

where R_{2K} represents the truncation error and is upper bounded by

$$R_{2K} = \sum_{k=K}^{\infty} P_{e_{2k}} \leqq \frac{(2K-3)!!}{(2K)!} \sqrt{4K-2} \cdot \frac{1}{2\sigma^{2k}} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{\left[\exp - \left(\dfrac{r^2(t_0)}{4\sigma^2}\right)\right] \cdot [\sum_{\ell \neq 0} |\, r(t_0 - \ell T)\, |]^{2K}}{\left[1 - \dfrac{1}{2K}\left\{\dfrac{\sum_{\ell \neq 0} |\, r(t_0 - \ell T)\, |}{\sigma}\right\}^2\right]},$$

$$= U_{2K} . \tag{12*}$$

Thus for a given truncation error bound, ϵ, we may always find a positive integer, K, such that

$$U_{2K} \leqq \epsilon. \tag{13}$$

For a real system, the truncation error is generally much smaller than ϵ. Therefore, fewer terms are needed in evaluating the error probability.

3.2 *Evaluation of Moments*

The series expansion of equation (9) can be readily evaluated if we can determine the moment, M_{2k}. The M_{2k}'s are given by

$$M_{2k} = \int_{\text{all } X} X^{2k}\, dF(X). \tag{14}$$

To evaluate M_{2k} according to equation (14) requires the knowledge of $dF(X)$; this is just as difficult to obtain as the evaluation of the error probability given by equation (8). However, we have found it possible to obtain a recurrence relation for M_{2k} by examining the first deriva-

* $(2K-3)!! = (2K-3) \cdot (2K-5) \cdots 3 \cdot 1.$

tive of the characteristic function. The recurrence formula makes the series expansion approach feasible, and is derived in Appendix B:

$$M_{2k} = -\left\{ \sum_{i=1}^{k} \binom{2k-1}{2i-1} (-1)^i M_{2(k-i)} f^{(2i-1)}(0) \right\}, \tag{15}$$

where

$$M_0 = 1 \tag{16}$$

$$f^{(2i-1)}(0) = \frac{2^{2i}(2^{2i}-1)}{2i} \mid B_{2i} \mid \sum_{\ell \neq 0} [r(t_0 - \ell T)]^{2i} \tag{17}$$

and B_{2i}'s are Bernoulli numbers.

3.3 *Truncated Pulse-Train Approximation*

For any real binary system, the message must be time-limited to a finite number of symbol durations, or we may even assume that $r(t)$ is time-limited to, say, N symbol durations. Thus the error probability may be calculated by evaluating the conditional error probability for each of 2^N possible data sequences and then averaging over all 2^N sequences. Since the number of possible data sequences grows exponentially with N, it would be impractical to evaluate the error probability by this straightforward method even with a digital computer. Hence, N must be confined to a small number; the error probability so obtained could at best be a poor approximation of the true error probability. However, in equation (9), the amount of computation involved grows only linearly with N. Therefore, the pulse train can be truncated at any desired point to assure a good approximation of the true channel.

IV. APPLICATIONS

The error probabilities for certain cases are calculated by equation (9) to determine the accuracy and the convergence of this new method.

4.1 *Case 1: Data Set* 203[7]

A 2400-baud DDD option of the Data Set 203 operating over a channel having symmetrical parabolic delay distortion, as shown in Fig. 2, is considered in this case. Tbe group delays at the carrier and the lower 3-dB frequencies are 0.6 ms relative to the center of the signal spectrum. The channel we considered is worse than a worst-case-C2 line. A 5-tap mean-square equalizer is used by the receiver to equalize the channel. A truncated 34-pulse-train approximation (19 samples after and 15 samples before the sampling instant t_0) for the

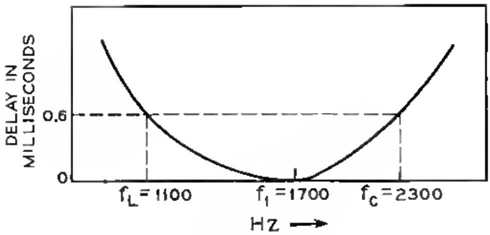

Fig. 2—Channel group-delay-frequency response.

equalized output impulse response was used. The equalized binary eye is about 70 percent open in this case. The input signal-to-noise ratio is 14 dB. The error probabilities at the equalizer output evaluated by equation (9) and the Chernoff inequality are shown in Fig. 3. Curve (a) is the Chernoff bound. Curve (b) is the error probability evaluated by taking a finite number of terms in equation (9). Curve (c) is the truncation error bound given by equation (12). It can be seen that taking the first nine terms in equation (9) assures less than one percent truncation error in evaluating the error probability. In this case, however, the actual series converges after only four terms. An improvement in accuracy by a factor of 15 is realized by this series expansion method compared to that obtained by Chernoff inequality.

4.2 *Case 2: Ideal Channel and Ideal Band-Limited Pulse*

The received pulse is assumed to have the form,

$$r(t) = \frac{\sin \pi t/T}{\pi t/T}. \tag{18}$$

The signal-to-noise ratio at the nominal sampling instant is taken to be 16 dB. In the absence of intersymbol interference, the system error probability is 10^{-10}. For a truncated 11-pulse-train approximation, the exact error probabilities and the error probabilities evaluated by taking a finite number of terms in equation (9) for different values of sampling instant and number of terms and equation (12) are shown in Figs. 4–5. It can be seen from these figures that the series converges more rapidly for smaller values of the quantity

$$q(t_0) = \left(\sum_{\substack{\ell=-5 \\ \ell \neq 0}}^{5} | r(t_0 - \ell T) | / \sigma \right)^2$$

[e.g., in this case, $q(0.05T) = 1.96$].

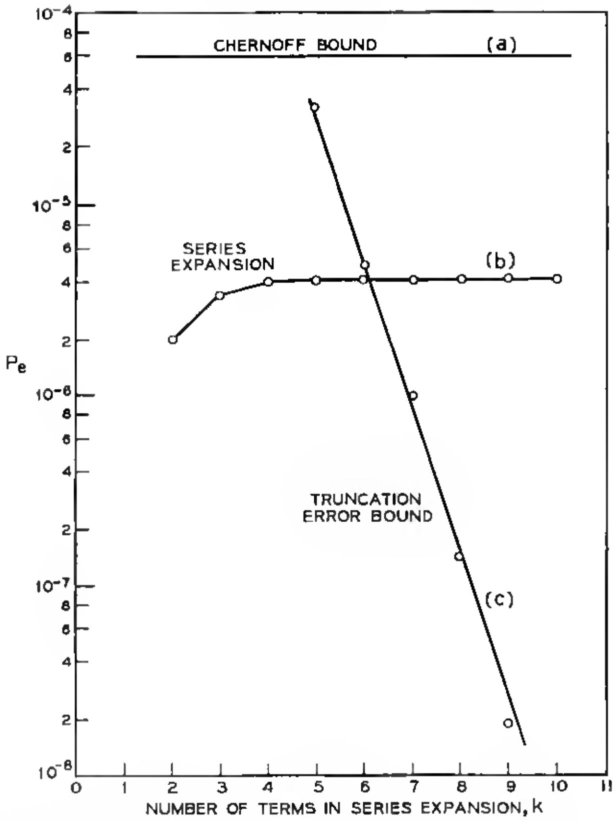

Fig. 3—Comparison of error probabilities obtained by Chernoff bound and series expansion method. $(S/N)_{\text{input}} = 14$ dB; data set 203 (2400-Baud Option); 5-tap mean square equalizer; parabolic delay distortion channel (see Fig. 2).

The series starts to oscillate when $q(t_0)$ is not small [e.g., $q(0.2T) = 30.8$]. At $t_0 = 0.2T$, the series did not converge well for the first eight terms in equation (9). However, it will converge to the exact value eventually. The error probabilities obtained by Chernoff bound,[5] exact calculation, and equation (9) are shown in Fig. 6. It is clear that this new alternative provides a significant improvement over the Chernoff bound.

4.3 *Case 3: Ideal Channel and Fourth-Order Chebyshev Pulse*[5]

In this case, a fourth-order Chebyshev filter is used. The received pulse is

$$r(t) = A_1 \cos(\omega_1 \mid t \mid /T - \Phi_1) \cdot \exp[-\alpha_1 \mid t \mid /T] + A_2 \cos(\omega_2 \mid t \mid /T - \Phi_2) \cdot \exp[-\alpha_2 \mid t \mid /T], \quad (19)$$

with

$$A_1 = 0.4023, \qquad A_2 = 0.7163,$$
$$\omega_1 = 2.839, \qquad \omega_2 = 1.176,$$
$$\Phi_1 = 0.7553, \qquad \Phi_2 = 0.1602,$$
$$\alpha_1 = 0.4587, \qquad \alpha_2 = 1.107.$$

The signal-to-noise ratio at the nominal sampling instant is taken to be 16 dB. For a truncated 11-pulse-train approximation, the exact error probabilities and the error probabilities obtained by taking a finite number of terms in equation (9) for various sampling instants and numbers of terms are shown in Figs. 7–8. The error probabilities obtained by the Chernoff[5] bound, the exact calculation, and equation (9) are shown in Fig. 9. The same results as in case 2 are observed.

V. SUMMARY AND CONCLUSIONS

In this study we have developed a new method of evaluating the error probability for synchronous data systems in the presence of intersymbol interference and additive gaussian noise under the fol-

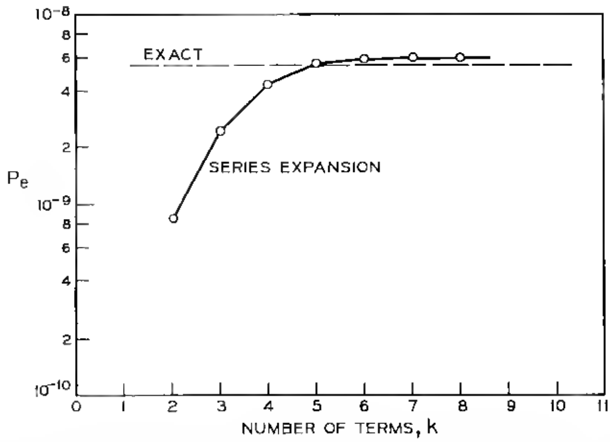

Fig. 4—Error probabilities versus number of terms in equation (9). Ideal band-limited signal. 11-pulse truncation approximation; sampling instant, $t = 0.05\ T$; (S/N) = 16 dB.

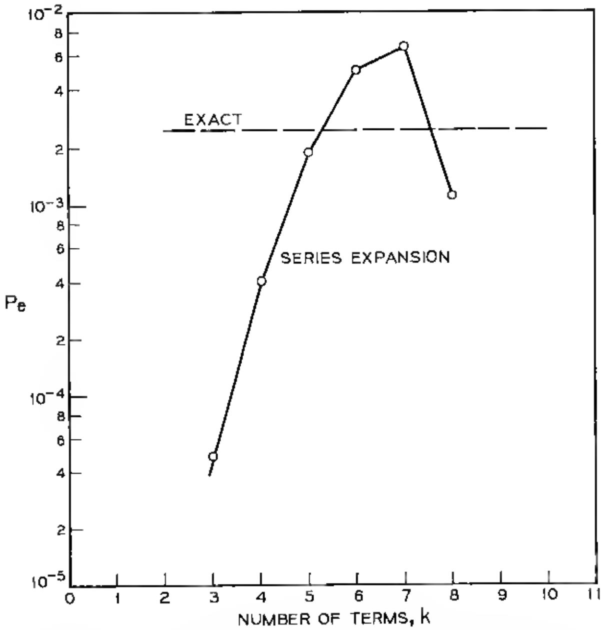

Fig. 5—Error probabilities versus number of terms in equation (9). Ideal band-limited signal. 11-pulse truncation approximation; sampling instant, $t = 0.2\ T$; $(S/N) = 16$ dB.

lowing assumptions. First, the information digits are identically and independently distributed. Second, the intersymbol interference converges absolutely. (For those pulses with absolutely divergent intersymbol interference, only finite truncated approximation of the real pulse will be used.) Three cases, which are representative of practical situations, are considered. The results show that this new method has a significant improvement in accuracy over Chernoff bound. For example, we consider the 2400-baud DDD option of the Data Set 203 operating over a channel having symmetrical delay distortion in excess of that of a worst-case C-2 line. A 5-tap mean-square equalizer is used by the receiver to equalize the channel. With a 14-dB input signal-to-noise ratio, the series expansion method provides a factor of 15 improvement over the Chernoff bound in estimating the error probability at the equalizer output.

The absolute convergence of the series expansion method is proved in Appendix A. An estimate of the terms required to reach the neighborhood of the true error probability is provided by equation (12). In

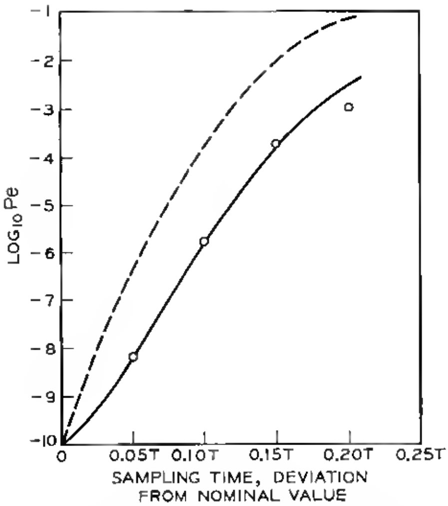

Fig. 6—Comparison of error probabilities obtained by Chernoff bound, exhaustive method, and series expansion method. Ideal band-limited signal (S/N) = 16 dB. [---Chernoff bound, ——— exhaustive method (11-pulse truncation), ooo series expansion (8-terms).]

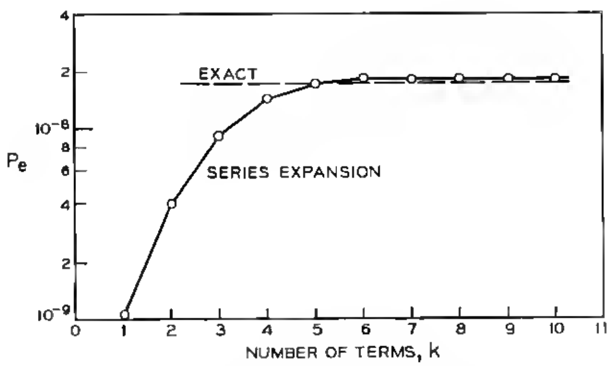

Fig. 7—Error probability versus number of terms in equation (9). Fourth-order Chebyshev pulse, 11-pulse truncation approximation; sampling instant, $t = 0.05\ T$;(S/N) = 16 dB.

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\begin{aligned}
a(t) = {} & A_1 \text{ COS } (\omega_1 |t|/T - \phi_1) \text{ EXP } (-\alpha_1 |t|/T) \\
& + A_2 \text{ COS } (\omega_2 |t|/T - \phi_2) \text{ EXP } (-\alpha_2 |t|/T).
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= 0.4023, & A_2 &= 0.7163, \\
\omega_1 &= 2.839, & \omega_2 &= 1.176, \\
\phi_1 &= 0.7553, & \phi_2 &= 0.1602, \\
\alpha_1 &= 0.4587, & \alpha_2 &= 1.107.
\end{aligned}
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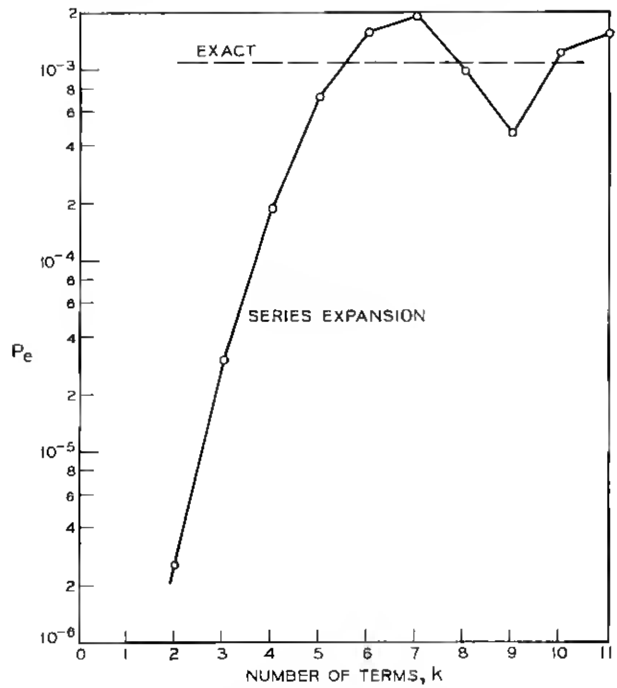

Fig. 8—Error probabilities versus number of terms in equation (9). Fourth-order Chebyshev pulse; 11-pulse truncation approximation; sampling instant, $t = 0.2\ T$; (S/N) = 16 dB.

actual systems, however, the true value is usually reached with only a small number of expansion terms. For example, in Fig. 3, the truncation error is less than 2×10^{-8} after taking into account the 9th term of the series expansion (which involves the 18th moment of the intersymbol interference); practically speaking, however, only three or four terms would be required for the series to converge in this example. In all the examples we considered, it is observed that a small error is assured by taking into account the first ten terms of the series.

The convergence is somewhat slower if the ratio of intersymbol interference to noise power $[q(t_0)]$ is large (see Section IV, case 2.), as indicated in Figs. 5 and 8. Under this condition, either the intersymbol interference is so bad that the system is not of practical interest, or the input signal-to-noise ratio is so high that the Chernoff bound already assures that the system performance is acceptable. For both cases, there is no need to evaluate the error probability.

For computation purposes every system must be approximated by a finite-memory-system. Since the computations involved in this new method increase only linearly with the length of the memory, a good approximation of the true channel may be obtained without excessive computation.

APPENDIX A

Convergence of the Series Expansion Method

In this Appendix, we shall prove that equation (9) is an absolutely convergent series. We know that

$$\begin{aligned} M_{2k} &= \int_{\text{all } X} X^{2k}\, dF(x) \\ &\leqq \int_{\text{all } X} (\sup X)^{2k}\, dF(x), \\ &= \{ \sum_{\ell \neq 0} | r(t_0 - \ell T) | \}^{2k} \end{aligned} \tag{20}$$

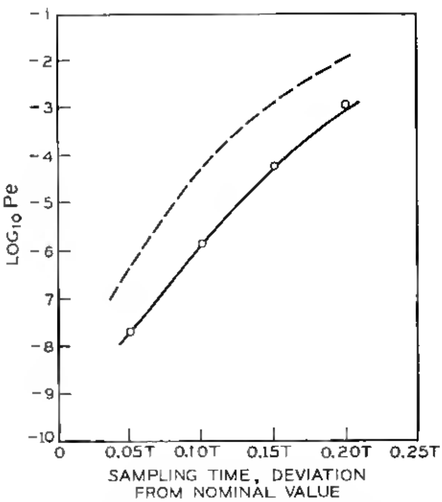

Fig. 9—Comparison of error probabilities obtained by Chernoff bound, exhaustive method, and series expansion method. Fourth-order Chebyshev pulse, (S/N) = 16 dB. [- - -Chernoff bound, ——— exhaustive method (11-pulse truncation), ooo series expansion (8-terms).]

and

$$H_{2K+1}(x) = (-1)^K 2^{K+\frac{1}{2}}[(2K-1)!!]\sqrt{2K+1} \cdot \exp(x^2/2) \cdot \left[\sin(\sqrt{4k+3}\,x) + O\left(\frac{1}{4\sqrt{K}}\right)\right]. \qquad (21)^*$$

Hence,

$$| P_{e_{2K}} | \leqq \frac{(2K-3)!!}{(2K)!} \sqrt{2K-1} \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{\sigma^2}\right)^K \frac{1}{\sqrt{\pi}} \cdot \exp\left[-\left(\frac{r^2(t_0)}{4\sigma^2}\right)\right] \cdot \{ \sum_{\ell \neq 0} | r(t_0 - \ell T) | \}^{2K}, = S_{2K} . \qquad (22)$$

The ratio of S_{2K+2} to S_{2K} is given by

$$\frac{S_{2K+2}}{S_{2K}} = \frac{\sqrt{2K-1}}{(2K+2)\sqrt{2K+1}} \left[\frac{\sum_{\ell \neq 0} | r(t_0 - \ell T) |}{\sigma}\right]^2 . \qquad (23)$$

For K sufficiently large, equation (23) is always less than unity. Therefore equation (9) is an absolutely convergent series.

APPENDIX B

Derivation of the Recurrence Relations for the Moment of Intersymbol Interference

It has been shown that the intersymbol interference converges absolutely to a random variable[6] X. The characteristic function of the random variable X is given by,

$$\Phi(\omega) = \int_{\text{all } X} e^{j\omega X}\, dF(X), = 1 + j\omega M_1 + \frac{(j\omega)^2}{2!} M_2 + \cdots + \frac{(j\omega)^k}{k!} M_k + \cdots . \qquad (24)$$

Therefore, we obtain

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* See Ref. 8.

$$\left.\begin{aligned} \Phi(0) &= 1 \\ \left.\frac{d^2\Phi(\omega)}{d\omega^2}\right|_{\omega=0} &= \Phi^2(0) = -M_2 , \\ &\vdots \\ \left.\frac{d^{2k}\Phi(\omega)}{d\omega^{2k}}\right|_{\omega=0} &= \Phi^{2k}(0) = (-1)^k M_{2k} , \\ &\vdots \end{aligned}\right\}. \tag{25}$$

Since a_ℓ's are identically and independently distributed random variables and with zero mean,

$$M_1 = M_3 = \cdots = M_{2k+1} = \cdots = 0 \quad \text{for} \quad k = 0, 1, 2, \cdots , \tag{26}$$

and

$$\Phi(\omega) = \prod_{\ell=1}^{N} \cos \omega r(t_0 - \ell T), \tag{27}$$

where a truncated N-pulse-train approximation of the channel impulse response is assumed.

The even-order moments could be obtained by differentiating equation (27) $2k$ times, but the right hand side expressions could become untractable. However, if we differentiate equation (27) once and regroup the terms, we obtain the following,

$$\begin{aligned} \Phi'(\omega) &= -\left[\sum_{\ell=1}^{N} r(t_0 - \ell T) \tan \omega r(t_0 - \ell T)\right]\cdot\Phi(\omega), \\ &= -f(\omega)\cdot\Phi(\omega). \end{aligned} \tag{28}$$

By successive differentiation of equation (28), a recurrence relation can now be obtained. Differentiating equation (28) $2k - 1$ times, we obtain

$$\Phi^{2k}(0) = -\left\{\sum_{i=1}^{k} \binom{2k-1}{2i-1} \Phi(0)^{2(k-i)} f^{2i-1}(0)\right\}, \tag{29}$$

where

$$f^{2i-1}(0) = \left.\frac{d^{2i-1}}{d\omega^{2i-1}} f(\omega)\right|_{\omega=0} . \tag{30}$$

The power series expansion of $\tan \omega r(t_0 - \ell T)$ around origin is

$$\tan \omega r(t_0 - \ell T) = \omega r(t_0 - \ell T) + \frac{(\omega r(t_0 - \ell T))^3}{3!} + \cdots$$
$$+ \frac{2^{2k}(2^{2k} - 1)}{(2k)!} \mid B_{2k} \mid (\omega r(t_0 - \ell T))^{2k-1} + \cdots , \qquad (31)$$

where B_{2k} is the Bernoulli number. It can be seen that

$$\frac{d^k}{d\omega^k} \tan \omega r(t_0 - \ell T) \Big|_{\omega=0} = [r(t_0 - \ell T)]^k \frac{2^{k+1}(2^{k+1} - 1)}{(k + 1)} \mid B_{k+1} \mid,$$
$$\text{for } k = \text{odd positive integers}, \qquad (32a)$$
$$= 0,$$
$$\text{for } k = \text{even positive integers}. \qquad (32b)$$

Thus,

$$f^k(0) = \frac{d^k}{d\omega^k} f(\omega) \Big|_{\omega=0} = \frac{2^{k+1}(2^{k+1} - 1)}{(k + 1)} \mid B_{k+1} \mid \cdot \lambda_{k+1} ,$$
$$\text{for } k = \text{odd positive integers}, \qquad (33a)$$
$$= 0,$$
$$\text{for } k = \text{even positive integers}. \qquad (33b)$$

where

$$\lambda_{k+1} = \sum_{\ell=1}^{N} [r(t_0 - \ell T)]^{k+1}. \qquad (33c)$$

Since

$$M_{2k} = (-1)^k \Phi^{2k}(0). \qquad (34)$$

Combining equations (34) and (29), we obtain the recurrence relation for M_{2k},

$$M_{2k} = -\left\{ \sum_{i=1}^{k} \binom{2k - 1}{2i - 1} (-1)^i M_{2(k-i)} f^{2i-1}(0) \right\} \qquad (35)$$

where $f^{2i-1}(0)$'s are given by equation (33a).

Knowing that $M_0 = 1$, all the higher order moments can be obtained via equation (35) without the knowledge of $dF(x)$.

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